Artin-Schreier presentations for Galois extensions of degree p^6 and maximal nilpotency class

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Background

At Omaha2020, Moles gave Artin-Schreier presentations for **all** nonabelian Galois extensions of degree p^4.

I will give Artin-Schreier presentations for Galois extensions of degree p^5 and p^6 with maximal nilpotency class.

The "Generalized Heisenberg" group

Heisenberg group:

$$\langle \sigma_1, \sigma_2, \sigma_3 : [\sigma_2, \sigma_1] = \sigma_3 \rangle$$

Generalized Heisenberg group:

$$H(p, n) = \langle \sigma_1, \ldots, \sigma_n : [\sigma_i, \sigma_1] = \sigma_{i+1} (2 \leq i < n) \rangle$$

These $H(p, n)$ groups appear in [Jam80] as $\Phi_2(1^3)$, $\Phi_3(1^4)$, $\Phi_9(1^5)$, and $\Phi_{35}(1^6)$.

Groups of Interest

Groups of order p^5 and p^6 $(p \geq 7)$ with maximal class [Jam80]:

$\Phi_9(1^5) \ = \langle \sigma_1, \ldots, \sigma_5 : [\sigma_i, \sigma_1] = \sigma_{i+1} \ (2 \leq i \leq 4) \rangle$

$\Phi_{10}(1^5) = \langle \sigma_1, \ldots, \sigma_5 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_2, \sigma_3] = \sigma_5 \ (2 \leq i \leq 4) \rangle$

$\Phi_{35}(1^6) = \langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1} \ (2 \leq i \leq 5) \rangle$

$\Phi_{36}(1^6) = \langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_2, \sigma_3] = \sigma_6 \ (2 \leq i \leq 5) \rangle$

$\Phi_{37}(1^6) = \langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_3, \sigma_4] = [\sigma_4, \sigma_2] = [\sigma_5, \sigma_2] = \sigma_6 \ (2 \leq i \leq 4) \rangle$

$\Phi_{38}(1^6) = \left\langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_2, \sigma_3] = \sigma_5^{-1}\sigma_6, [\sigma_2, \sigma_4] = \sigma_6 \ (2 \leq i \leq 5) \right\rangle$

$\Phi_{39}(1^6) = \langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_3, \sigma_4] = [\sigma_4, \sigma_2] = [\sigma_5, \sigma_2] = \sigma_6, [\sigma_2, \sigma_3] = \sigma_5 \ (2 \leq i \leq 4) \rangle *$

Since $p \geq 7$, each generator has order p.

Groups of Interest

James published *The Groups of Order p^6 (p an odd prime)* in 1980, with

$$\Phi_{39}(1^6) = \langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_3, \sigma_4] = [\sigma_4, \sigma_2] = [\sigma_5, \sigma_2] = \sigma_6, [\sigma_2, \sigma_3] = \sigma_5 \ (2 \leq i \leq 4) \rangle$$

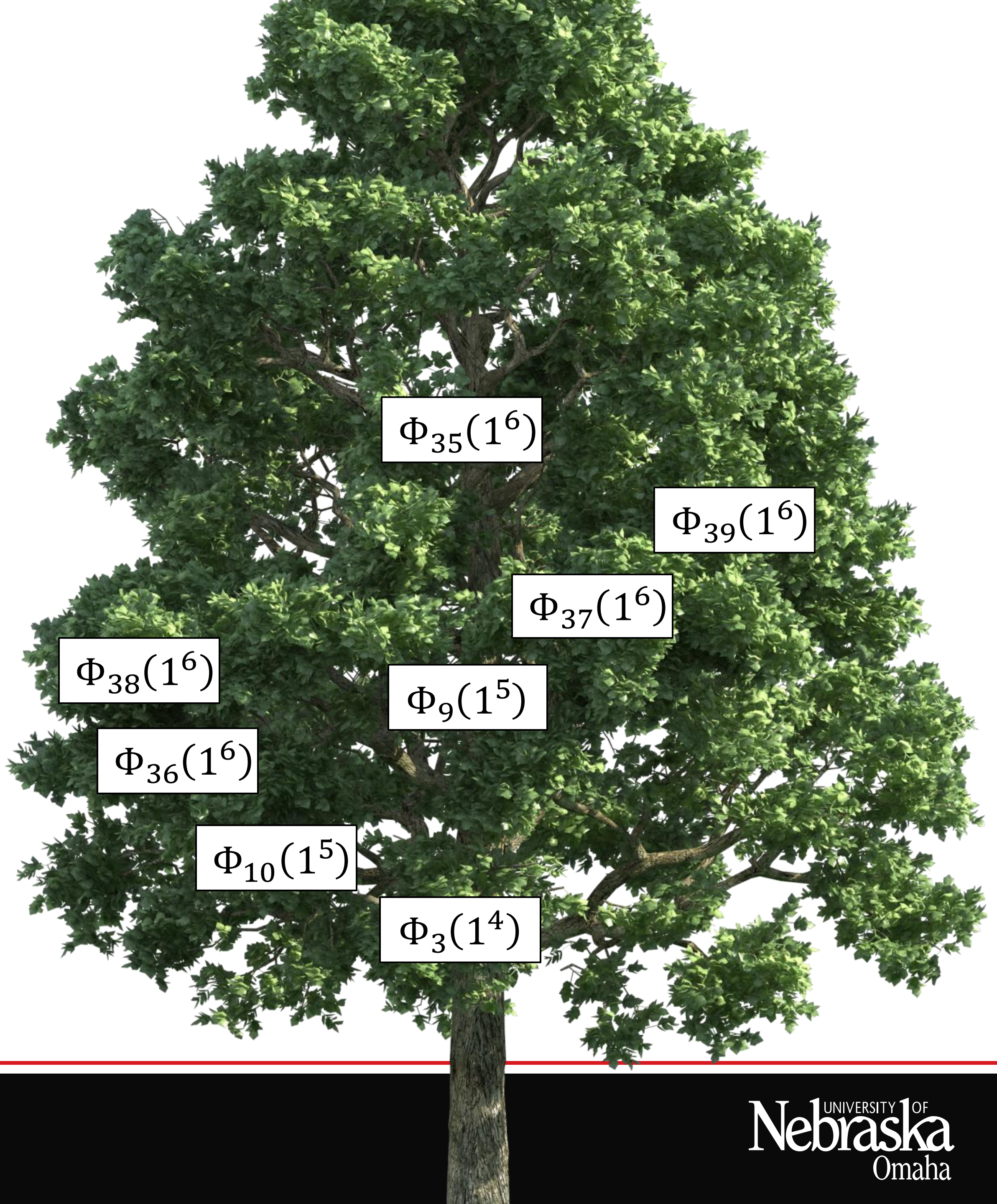
Newman et. al. published *Presentations for Groups of Order p^6 for* $p \geq 7$ in 2023, with

$$\Phi_{39}(1^6) = \langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_3, \sigma_4] = [\sigma_4, \sigma_2] = [\sigma_2, \sigma_5] = \sigma_6, [\sigma_2, \sigma_3] = \sigma_5 \ (2 \leq i \leq 4) \rangle$$

This has a slight effect on my presentation of $\Phi_{39}(1^6)$ as a Galois group.

Groups of Interest

- $\Phi_9(1^5) = H(p,5)$
- $\Phi_{10}(1^5)/\langle\sigma_5\rangle \approx H(p,4)$
- $\Phi_{35}(1^6) = H(p,6)$
- $\Phi_{36}(1^6)/\langle\sigma_6\rangle \approx \Phi_9(1^5)$
- $\Phi_{37}(1^6)/\langle\sigma_6\rangle \approx \Phi_9(1^5)$
- $\Phi_{38}(1^6)/\langle\sigma_6\rangle \approx \Phi_{10}(1^5)$
- $\Phi_{39}(1^6)/\langle\sigma_6\rangle \approx \Phi_{10}(1^5)$.



Artin-Schreier Theory

Let $G = Gal(N/K)$, where N has characteristic p, and denote $\wp(x) = x^p - x$. Consider an extension M/K with degree n and an elementary abelian Galois group:

$$Gal(M/K) = \langle \sigma_1, \dots, \sigma_n : \sigma_i \in G, \sigma_i^p = 1 \rangle \approx C_p^n$$

- Then $M = K(\alpha_1, \dots, \alpha_n)$ where α_i satisfy the irreducible Artin-Schreier polynomials $\wp(\alpha_i) = \beta_i$ for $\beta_i \in K$.
- Also, $(\sigma_i - 1)\alpha_j = 0$ for $i \neq j$ and WLOG $(\sigma_i - 1)\alpha_i = 1$. So, we say $(\sigma_i - 1)\alpha_j = \delta_{ij}$.

Generalized Heisenberg group

Let $G = H(p,n) = Gal(N/K)$. Notice $G/\langle\sigma_n\rangle \approx C_p^{n-1}$, so N^{σ_n} is a Galois extension, which we call M. Then $M = K(\alpha_1)$ and

- $\wp(\alpha_1) = \gamma_1 \in K$, $\wp(\alpha_i) = \beta_i \in M$ $(i \in \{2, \dots, n\})$
- $(\sigma_1 - 1)\alpha_1 = 1$, $(\sigma_i - 1)\alpha_j = \delta_{ij}$, $(\sigma_i - 1)\alpha_1 = 0$ $(i, j \in \{2, \dots, n\})$

We want to find $(\sigma_1 - 1)\alpha_i$, then write $\wp(\alpha_i)$ in terms of constants in K.

Generalized Heisenberg group

Group actions:

$$(\sigma_1 - 1)\alpha_i = \alpha_{i-1} \ (3 \leq i \leq n)$$
$$(\sigma_i - 1)\alpha_j = \delta_{ij} \ (i, j \ otherwise)$$

Artin-Schreier polynomials:

$$\wp(\alpha_1) = \gamma_1 \in K$$
$$\wp(\alpha_i) = \sum_{m=0}^{i-2} \gamma_{i-m} \binom{\alpha_1}{m}, \ \gamma_i \in K \ (2 \leq i \leq n)$$

Using Additive Hilbert's Theorem 90

Let N/K be a Galois extension with $Gal(N/K) = G$ and H normal in G. Set $M = N^H$ and consider the generators of N/M, namely $\alpha_1, \dots, \alpha_k$ such that $N = M(\alpha_1, \dots, \alpha_k)$ for some k. Given $\sigma \in G/H$ and some α_i, then $(\sigma - 1)\alpha_i = A \in N$ and M/M^σ is a cyclic extension.

Consider $B \in N$ such that $A - B \in M$. Then by AHT90, if $\mathrm{Tr}_{M/M^\sigma}(A - B) = 0$, there is some $\alpha_i' \in N$ such that

- $(\tau - 1)\alpha_i = (\tau - 1)\alpha_i{}'$ for all $\tau \in H$,
- $\wp(\alpha_i') \in L$ when $\wp(\alpha_i) \in L$

Then we can replace the generator α_i with $\alpha_i{}'$ and $(\sigma - 1)\alpha_i{}' = B$. Also, we maintain previous group actions.

Example: Classifying $\Phi_{36}(1^6)$

Let $\mathrm{G} = \Phi_{36}(1^6) = \langle \sigma_1, \ldots, \sigma_6 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_2, \sigma_3] = \sigma_6 \rangle = Gal(N/K)$ for $2 \leq i \leq 5$. Notice $\langle \sigma_3, \ldots, \sigma_6 \rangle \approx C_p^4$ and normal in G, so $G/\langle \sigma_3, \ldots, \sigma_6 \rangle$ is a Galois group. Let $M = N^{\langle \sigma_3, \ldots, \sigma_6 \rangle}$ and construct the A-S polynomials:

$$\wp(\alpha_i) \in \mathrm{M}\ (3 \leq \mathrm{i} \leq 6), \qquad \wp(\alpha_i) = \gamma_{\mathrm{i}} \in \mathrm{K}\ (1 \leq \mathrm{i} \leq 2)$$

With the following group actions:

$$(\sigma_i - 1)\alpha_i = 0\ \ (3 \leq i \leq 6, 1 \leq j \leq 2)$$
$$(\sigma_i - 1)\alpha_j = \delta_{ij}\ \ (i, j \in \{1,2\}\ or\ i, j \in \{3, \ldots, 6\})$$

Example: Classifying $\Phi_{36}(1^6)$

Since $G/\langle\sigma_6\rangle \approx \Phi_9(1^5) = H(p,5)$, we have the following group actions and polynomials:

$$(\sigma_1 - 1)\alpha_i = \alpha_{i-1}\ (3 \leq i \leq 5)$$
$$(\sigma_i - 1)\alpha_j = \delta_{ij}\ (i,j \in \{1,\ldots 5\}\ otherwise)$$

$$\wp(\alpha_1) = \gamma_1 \in K$$
$$\wp(\alpha_i) = \sum_{m=0}^{i-2} \gamma_{i-m} \binom{\alpha_1}{m},\ \gamma_i \in K\ (2 \leq i \leq 5)$$

Example: Classifying $\Phi_{36}(1^6)$

Group Actions: σ_2 on α_6

- Let $(\sigma_2 - 1)\alpha_6 = A_2$. Since $\sigma_4, \sigma_5, \sigma_6$ commute with σ_2, $A_2 \in N^{\langle \sigma_4, \sigma_5, \sigma_6 \rangle}$
- Apply $[\sigma_2, \sigma_3] = \sigma_6$ to α_6 to get $(\sigma_3 - 1)(A_2 + \alpha_3) = 0$
- Since $\alpha_3 \in N^{\langle \sigma_4, \sigma_5, \sigma_6 \rangle}$, we have $A_2 + \alpha_3 \in N^{\langle \sigma_3, \sigma_4, \sigma_4, \sigma_6 \rangle} = M$
- Notice $\mathrm{Tr}_{M/M^{\sigma_2}}(A_2 + \alpha_3) = 0$. By AHT90 $(\sigma_2 - 1)\alpha_6 = -\alpha_3$

Example: Classifying $\Phi_{36}(1^6)$

Group Actions: σ_1 on α_6

- Let $(\sigma_1 - 1)\alpha_6 = A_1$. Since $[\sigma_1, \sigma_6] = 1$, $A_1 \in N^{\sigma_6}$
- Apply $[\sigma_5, \sigma_1] = \sigma_6$ to α_6 to get $(\sigma_5 - 1)(A_1 - \alpha_5) = 0$
- Since $\alpha_5 \in N^{\sigma_6}, A_1 - \alpha_5 \in N^{\langle\sigma_5,\sigma_6\rangle}$
- Apply $[\sigma_4, \sigma_1] = \sigma_5$ and $[\sigma_3, \sigma_1] = \sigma_4$ to α_6. Since $\alpha_6 \in N^{\langle\sigma_3,\sigma_4,\sigma_5\rangle}, A_1 \in N^{\langle\sigma_3,\sigma_4\rangle}$
- Since $\alpha_5 \in N^{\langle\sigma_3,\sigma_4\rangle}, A_1 - \alpha_5 \in N^{\langle\sigma_3,\sigma_4,\sigma_5,\sigma_6\rangle}$
- Apply $[\sigma_2, \sigma_1] = \sigma_3$ to α_6 to get $(\sigma_2 - 1)\left(A_1 + \binom{\alpha_2}{2}\right) = 0$
- Since $\alpha_2 \in N^{\langle\sigma_3,\sigma_4,\sigma_5,\sigma_6\rangle}$ and $\alpha_5 \in N^{\langle\sigma_2\rangle}$, $A_1 - \alpha_5 + \binom{\alpha_2}{2} \in N^{\langle\sigma_2,\sigma_3,\sigma_4,\sigma_5,\sigma_6\rangle} = K(\alpha_1)$
- Notice $\mathrm{Tr}_{K(\alpha_1)/K}\left(A_1 - \alpha_5 + \binom{\alpha_2}{2}\right) = 0$. By AHT90, $(\sigma_1 - 1)\alpha_6 = \alpha_5 - \binom{\alpha_2}{2}$

Example: Classifying $\Phi_{36}(1^6)$

Artin-Schreier Polynomials: $\wp(\alpha_6)$

- $\wp$ is a linear map that commutes with every generator, so

$$(\sigma_1 - 1)\wp(\alpha_6) = \wp(\alpha_5) - \wp\left(\binom{\alpha_2}{2}\right) = \gamma_5 + \gamma_4\alpha_1 + \gamma_3\binom{\alpha_1}{2} + \gamma_2\binom{\alpha_1}{3} - \gamma_2\alpha_2 - \binom{\gamma_2}{2}$$

$$(\sigma_2 - 1)\wp(\alpha_6) = -\wp(\alpha_3) = -\gamma_3 - \gamma_2\alpha_1$$

- Since $(\sigma_1 - 1)\binom{\alpha_1}{k} = \binom{\alpha_1}{k-1}$ and σ_1 fixes α_2,

$$(\sigma_1 - 1)\left(\wp(\alpha_6) + \left(\binom{\gamma_2}{2} - \gamma_5\right)\alpha_1 - \gamma_4\binom{\alpha_1}{2} - \gamma_3\binom{\alpha_1}{3} - \gamma_2\binom{\alpha_1}{4} + \gamma_2\alpha_2\alpha_1\right) = 0$$

- Next, compute $(\sigma_2 - 1)$ of the above element:

$$(\sigma_2 - 1)\left(\wp(\alpha_6) + \left(\binom{\gamma_2}{2} - \gamma_5\right)\alpha_1 - \gamma_4\binom{\alpha_1}{2} - \gamma_3\binom{\alpha_1}{3} - \gamma_2\binom{\alpha_1}{4} + \gamma_2\alpha_2\alpha_1\right) = -\gamma_3$$

Example: Classifying $\Phi_{36}(1^6)$

Artin-Schreier Polynomials: $\wp(\alpha_6)$

$$(\sigma_2 - 1)(\wp(\alpha_6) + (\binom{\gamma_2}{2} - \gamma_5)\alpha_1 - \gamma_4\binom{\alpha_1}{2} - \gamma_3\binom{\alpha_1}{3} - \gamma_2\binom{\alpha_1}{4} + \gamma_2\alpha_2\alpha_1 + \gamma_3\alpha_2) = 0$$

- σ_1 and σ_2 fix an element in M, so this element equals some $\gamma_6 \in K$:

$$\wp(\alpha_6) - \sum_{m=0}^{4} \gamma_{6-m}\binom{\alpha_1}{m} + \binom{\gamma_2}{2}\alpha_1 + \gamma_2\alpha_2\alpha_1 + \gamma_3\alpha_2 = \gamma_6$$

- This gives us our remaining Artin-Schreier polynomial:

$$\wp(\alpha_6) = \sum_{m=0}^{4} \gamma_{6-m}\binom{\alpha_1}{m} + \binom{\gamma_2}{2}\alpha_1 + \gamma_2\alpha_2\alpha_1 + \gamma_3\alpha_2 : \gamma_i \in K$$

Overview

- $\Phi_{37}(1^6)$ is very similar to $\Phi_{36}(1^6)$, except $\langle \sigma_3, \dots, \sigma_6 \rangle$ is not normal in $\Phi_{37}(1^6)$, while $\langle \sigma_4, \sigma_5, \sigma_6 \rangle$ is
- $\Phi_{38}(1^6)$ and $\Phi_{39}(1^6)$ are similar to $\Phi_{36}(1^6)$ and $\Phi_{37}(1^6)$, respectively, except $\Phi_{36}(1^6)/\langle \sigma_6 \rangle \approx \Phi_{37}(1^6)/\langle \sigma_6 \rangle \approx \Phi_9(1^5)$, while $\Phi_{38}(1^6)/\langle \sigma_6 \rangle \approx \Phi_{39}(1^6)/\langle \sigma_6 \rangle \approx \Phi_{10}(1^5)$
- Since $\Phi_{38}(1^6)$ and $\Phi_{39}(1^6)$ both require $\Phi_{10}(1^5)$, so we need one more classification

Classifying $\Phi_{10}(1^5)$

Let $G = \Phi_{10}(1^5) = \langle \sigma_1, \ldots, \sigma_5 : [\sigma_i, \sigma_1] = \sigma_{i+1}, [\sigma_2, \sigma_3] = \sigma_5 \rangle = Gal(N/K)$ for $2 \leq i \leq 4$. Notice $\langle \sigma_3, \sigma_4, \sigma_5 \rangle \approx C_p^3$ and normal in G, so $G/\langle \sigma_3, \sigma_4, \sigma_5 \rangle$ is a Galois group. Let $M = N^{\langle \sigma_3, \sigma_4, \sigma_5 \rangle}$ and construct the A-S polynomials:

$$\wp(\alpha_i) \in \mathrm{M}\ (3 \leq \mathrm{i} \leq 5), \qquad \wp(\alpha_i) = \gamma_{\mathrm{i}} \in \mathrm{K}\ (1 \leq \mathrm{i} \leq 2)$$

With the following group actions:

$$(\sigma_i - 1)\alpha_i = 0\ \ (3 \leq i \leq 5\ 1 \leq j \leq 2)$$

$$(\sigma_i - 1)\alpha_j = \delta_{ij}\ \ (i, j \in \{1,2\}\ or\ i, j \in \{3, \ldots, 5\})$$

Since $G/\langle \sigma_5 \rangle \approx H(p, 4)$, we have additional group actions and polynomials.

Classifying $\Phi_{10}(1^5)$

Group actions:

$$(\sigma_1 - 1)\alpha_5 = \alpha_4 - \binom{\alpha_2}{2} \qquad (\sigma_1 - 1)\alpha_i = \alpha_{i-1}\ (3 \le i \le 4)$$

$$(\sigma_2 - 1)\alpha_5 = -\alpha_3 \qquad (\sigma_i - 1)\alpha_j = \delta_{ij}\ (i, j\ otherwise)$$

Artin-Schreier polynomials:

$$\wp(\alpha_1) = \gamma_1 \in K$$

$$\wp(\alpha_i) = \sum_{m=0}^{i-2} \gamma_{i-m} \binom{\alpha_1}{m},\ (2 \le i \le n)$$

$$\wp(\alpha_5) = \sum_{m=0}^{3} \gamma_{5-m} \binom{\alpha_1}{m} - \binom{\alpha_2}{2} \alpha_1 - \gamma_2 \alpha_1 \alpha_2 - \gamma_3 \alpha_2$$

Comparing Group Actions of p^6 Galois groups

$\Phi_{36}(1^6)$:

$$(\sigma_1 - 1)\alpha_i = \alpha_{i-1},$$
$$(\sigma_1 - 1)\alpha_6 = \alpha_5 - \binom{\alpha_2}{2},$$
$$(\sigma_2 - 1)\alpha_6 = -\alpha_3.$$

$(3 \leqslant i \leqslant 5)$

$\Phi_{37}(1^6)$:

$$(\sigma_1 - 1)\alpha_i = \alpha_{i-1},$$
$$(\sigma_1 - 1)\alpha_6 = -\binom{\alpha_3}{2},$$
$$(\sigma_2 - 1)\alpha_6 = \alpha_4 + \alpha_5,$$
$$(\sigma_3 - 1)\alpha_6 = -\alpha_4.$$

$(3 \leqslant i \leqslant 5)$

$\Phi_{38}(1^6)$:

$$(\sigma_1 - 1)\alpha_i = \alpha_{i-1},$$
$$(\sigma_1 - 1)\alpha_6 = \alpha_5 + \binom{\alpha_2}{2},$$
$$(\sigma_2 - 1)\alpha_6 = \alpha_3 - \alpha_4,$$
$$(\sigma_1 - 1)\alpha_5 = \alpha_4 - \binom{\alpha_2}{2},$$
$$(\sigma_2 - 1)\alpha_5 = -\alpha_3.$$

$(3 \leqslant i \leqslant 4)$

$\Phi_{39}(1^6)$:

$$(\sigma_1 - 1)\alpha_i = \alpha_{i-1},$$
$$(\sigma_1 - 1)\alpha_6 = -\binom{\alpha_3}{2} - \binom{\alpha_2}{3},$$
$$(\sigma_2 - 1)\alpha_6 = \alpha_4 + \alpha_5,$$
$$(\sigma_3 - 1)\alpha_6 = -\alpha_4,$$
$$(\sigma_1 - 1)\alpha_5 = \alpha_4 - \binom{\alpha_2}{2},$$
$$(\sigma_2 - 1)\alpha_5 = -\alpha_3.$$

$(3 \leqslant i \leqslant 4)$

Comparing A-S Polynomials of p^6 Galois groups

$\Phi_{36}(1^6)$:

$$\wp(\alpha_6) = \sum_{m=0}^{4} \gamma_{6-m}\binom{\alpha_1}{m} + \wp\left(\binom{\alpha_2}{2}\right)\alpha_1 + \gamma_3\alpha_2 = \sum_{m=0}^{4} \gamma_{6-m}\binom{\alpha_1}{m} + \binom{\gamma_2}{2}\alpha_1 + \wp(\alpha_3)\alpha_1$$

$\Phi_{37}(1^6)$:

$$\begin{aligned}\wp(\alpha_6) = &\Big(\gamma_6 - \gamma_4\alpha_3 + \gamma_5\alpha_2 + \gamma_4\alpha_2\Big) + \Big(- \gamma_3\alpha_3 + \gamma_4\alpha_2 + \gamma_3\alpha_2 - \binom{\gamma_3}{2}\Big)\alpha_1 \\ &+ \Big(- \gamma_2\alpha_3 + \gamma_3\alpha_2 + \gamma_2\alpha_2 - \gamma_2\gamma_3 - \binom{\gamma_2}{2}\Big)\binom{\alpha_1}{2} + \Big(+ \gamma_2\alpha_2 - \gamma_2^2\Big)\binom{\alpha_1}{3}.\end{aligned}$$

$\Phi_{38}(1^6)$:

$$\wp(\alpha_6) = \sum_{m=0}^{4} \gamma_{6-m}\binom{\alpha_1}{m} + \binom{\gamma_2}{2}\alpha_1 - \binom{\gamma_2}{2}\binom{\alpha_1}{2} + (\gamma_3 - \gamma_4)\alpha_2 + (\gamma_2 - \gamma_3)\alpha_1\alpha_2 - \gamma_2\binom{\alpha_1}{2}\alpha_2$$

$\Phi_{39}(1^6)$:

$$\begin{aligned}\wp(\alpha_6) = &\gamma_6 - \gamma_4\alpha_3 - \gamma_3\alpha_1\alpha_3 - \gamma_2\binom{\alpha_1}{2}\alpha_3 + \gamma_5\alpha_2 + \gamma_4(\alpha_1 + 1)\alpha_2 + \gamma_3\binom{\alpha_1+1}{2}\alpha_2 + \gamma_2\binom{\alpha_1+1}{3}\alpha_2 \\ &- \binom{\gamma_2}{2}(\alpha_1 + 1)\alpha_2 - \gamma_3\binom{\alpha_2}{2} + \gamma_2\binom{\alpha_2}{2} + \binom{\gamma_2}{2}\alpha_2 - \left(\binom{\gamma_3}{2} + \binom{\gamma_2}{2}\right)\alpha_1 - \left(\gamma_2\gamma_3\binom{\gamma_2}{2}\right)\binom{\alpha_1}{2} - \gamma_2^2\binom{\alpha_1}{3}\end{aligned}$$

Next steps

- Simplify polynomials
- Find connections between Galois group presentations
- Find explicit relation between commutators and group actions

Thank you for your attention!